

# Reflexive cones<sup>\*</sup>

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## Abstract

Reflexive cones in Banach spaces are cones with weakly compact intersection with the unit ball. In this paper we study the structure of this class of cones. We investigate the relations between the notion of reflexive cones and the properties of their bases. This allows us to prove a characterization of reflexive cones in term of the absence of a subcone isomorphic to the positive cone of  $\ell_1$ . Moreover, the properties of some specific classes of reflexive cones are investigated. Namely, we consider the reflexive cones such that the intersection with the unit ball is norm compact, those generated by a Schauder basis and the reflexive cones regarded as ordering cones in Banach spaces. Finally, it is worth to point out that a characterization of reflexive spaces and also of the Schur spaces by the properties of reflexive cones is given.

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## 1 Introduction

The study of cones is central in many fields of pure and applied mathematics. In Functional Analysis, the theory of partially ordered spaces and Riesz spaces

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is based on the properties of cones, how these properties are related with the algebraic and topological properties of the spaces, the structure of linear operators, etc. In Mathematical economics (see [2, 4]) the theory of partially ordered spaces is used in General Equilibrium Theory and in finance the lattice structure is necessary for the representation of the different kinds of derivatives. Also the geometry of ordering cones is crucial in the theory of vector optimization, see in [9] and the reference therein.

Our motivation for this article was to study the class of cones  $P$  of a Banach spaces  $X$  which coincide with their second dual cone in  $X^{**}$ , i.e.  $P = P^{00}$  and we had called these cones reflexive.

In the first steps of our work, Section 3, we proved that  $P$  is reflexive if and only if the intersection  $B_X^+ = P \cap B_X$  of  $P$  with the unit ball  $B_X$  of  $X$  is weakly compact. Since this property seems more accessible and more natural, we decided to start by this property as definition of reflexive cones. Based on the above characterization of reflexive cones we give, Theorem 3.5, a characterization of reflexive spaces.

We remark that in Banach spaces, cones with weakly compact  $B_X^+$  (reflexive cones in the present terminology) have been studied in [19] and the results of this article are applied in economic models. Also in [6] structural properties of cones related with this kind of cones are given.

In Section 4 we continue the study of [19] and [6] by studying the bases of reflexive cones. The relationships with the existence of bounded and unbounded base of a reflexive cone allow us to prove that a closed cone  $P$  of a Banach space is reflexive if and only if  $P$  does not contain a closed cone isomorphic to the positive cone  $\ell_1^+$  of  $\ell_1$ , Theorem 4.5. Note that necessary and sufficient conditions in order a closed cone of a Banach space to be isomorphic to the positive cone of  $\ell_1$  are given in [17].

Moreover, it is worth pointing out that the existence of a basic sequence inside a reflexive cone, Theorem 4.7 and 4.5, depends on the existence of a bounded or an unbounded base of the cone.

During our study we found interesting examples of cones with norm compact positive part  $B_X^+$  of the unit ball and we called these cones strongly reflexive. Section 5 is devoted to the study of this class of reflexive cones. (We start by mentioning an old result of Klee, Theorem 5.3.)

We prove a characterization of Schur spaces as the Banach spaces where every reflexive cone with a bounded base is strongly reflexive, Theorem 5.6.

We give also different examples of reflexive cones. Especially if  $X$  is a Banach lattice with a positive Schauder basis then we prove that a strongly reflexive cone  $P \subseteq X_+$  exists so that the subspace  $Y = P - P$  generated by  $P$  is dense in  $X$  and we give a method for the determination of this cone, Theorem 5.7.

Also in Example 5.9 a strongly reflexive cone  $P \subseteq L_1^+[0, 1]$  and in Example 3.7 a reflexive cone  $P \subseteq L_1^+[0, 1]$  are determined so that, in both cases, the subspace  $Y = P - P$  generated by  $P$  is dense in  $L_1[0, 1]$ .

We close Section 5 by proving that any positive operator from a Banach lattice into a Banach space ordered by a reflexive (strongly reflexive) pointed cone is weakly compact(compact), see Theorem 5.11. Although the proof of

this result is easy, combined with our examples of reflexive cones can determine different classes of compact operators.

In Section 6 we give a characterization of the positive cone  $P$  of a Schauder basis  $\{x_n\}$  of a Banach space  $X$ , in terms of the properties of the basis itself, in the same spirit of the classical result of James, [12]. We show, Theorem 6.2 that if the cone  $P$  is reflexive, then  $\{x_n\}$  is boundedly complete on  $P$  and in Theorem 6.3 we prove that if the basis  $\{x_n\}$  is shrinking and boundedly complete on  $P$ , then  $P$  is reflexive. We show also, Example 6.4, that the assumption "the shrinking basis  $\{x_n\}$  is boundedly complete only on the cone  $P$ ", is not enough to ensure the reflexivity of the whole space  $X$ .

In Section 7 we suppose  $X$  is a Banach space ordered by a reflexive cone  $P$  and we study order properties of  $X$ . First we show that if  $P$  is normal then  $X$  is order complete, Theorem 7.1. In the sequel we study the lattice property. By accompanying the basic result of [17], where characterizations of the positive cone  $\ell_1^+$  of  $\ell_1$  are given and our result that  $P$  as a reflexive cone does not contains  $\ell_1^+$ , we prove that if  $P$  has a bounded base, then  $P$  cannot be a lattice cone (for the exact assumptions see in Theorem 7.3 and its corollaries).

Finally, we recall that in the Choquet theory, cones  $P$  in locally convex Hausdorff spaces  $X$  with a compact base are studied. As we note in Remark 7.6, our results are to an other direction and independent of the ones of this theory.

## 2 Notations and preliminaries

In this article we will denote by  $X$  a Banach space, by  $X^*$  the norm dual of  $X$  and by

$$B_X = \{x \in X : \|x\| \leq 1\},$$

the closed unit ball of  $X$ .

A nonempty, convex subset  $P$  of  $X$  is a **cone** if  $\lambda P \subseteq P$  for every real number  $\lambda \geq 0$ . If in addition,  $P \cap (-P) = \{0\}$ ,  $P$  is a **pointed** or a **proper** cone. Note that in the literature, instead of the terms cone and pointed cone the names wedge and cone are often used.

For any  $A \subseteq X$  we denote by  $\overline{A}$ , the closure of  $A$ , by  $\text{int}A$  the interior of  $A$  by  $\text{co}(A)$  the convex hull and by  $\overline{\text{co}}(A)$  the closed convex hull of  $A$ . Also we denote by  $\text{cone}(A)$  ( $\overline{\text{cone}}(A)$ ) the smallest cone, (closed cone) containing  $A$ . By  $\overline{A}^w$  we denote the closure of  $A$  in the weak topology and by  $\overline{A}^{w*}$  the closure of  $A$  in the weak-star topology, whenever  $A$  is a subset of a dual space. If the set  $A$  is convex, we have

$$\text{cone}(A) = \{\lambda a : a \in A, \lambda \geq 0\}.$$

If the set  $A$  is closed and bounded, it is easy to show that  $\text{cone}(A)$  is closed.

Suppose that  $P$  is a cone of  $X$ .  $P \subseteq X$  induces the partial ordering  $\leq_P$  in  $X$  so that  $x \leq_P y$  if and only if  $y - x \in P$ , for any  $x, y \in X$ ; in the sequel, for the sake of simplicity, we will use the symbol  $\leq$  instead of  $\leq_P$  whenever no confusion can arise. This order relation is antisymmetric if and only if  $P$

is pointed. If  $x, y \in X$  with  $x \leq y$ , the set  $[x, y] = \{z \in X \mid x \leq z \leq y\}$  is the **order interval** defined by  $x, y$ . If  $P - P = X$  the cone  $P$  is **generating**. The cone  $P$  gives an **open decomposition of  $X$**  if there exists  $\rho > 0$  so that  $\rho B_X \subseteq B_X^+ - B_X^+$ , where  $B_X^+ = B_X \cap P$ . In Banach spaces any closed and generating cone gives an open decomposition, [13], Theorem 3.5.2. The cone  $P$  is **normal** if there exists  $c \in \mathbb{R}$  so that for any  $x, y \in X$ ,  $0 \leq x \leq y$  implies  $\|x\| \leq c \|y\|$ .

A linear functional  $f$  of  $X$  is said **positive** (on  $P$ ) if  $f(x) \geq 0$  for each  $x \in P$  and **strictly positive** (on  $P$ ) if  $f(x) > 0$  for each  $x \in P, x \neq 0$ . The set of the continuous positive functionals on the cone  $P$  is a cone named the **dual** (or **polar**) cone of  $P$  and it is denoted by

$$P^0 = \{x^* \in X^* : x^*(x) \geq 0 \text{ for each } x \in P\}.$$

If a strictly positive linear functional exists, the cone  $P$  is pointed. A convex subset  $B$  of  $P$  is a **base for the cone  $P$**  if for each  $x \in P, x \neq 0$  a unique real number  $f(x) > 0$  exists such that  $\frac{x}{f(x)} \in B$ . Then the function  $f$  is additive and positively homogeneous on  $P$  and  $f$  can be extended to a linear functional on  $P - P$  by the formula  $f(x_1 - x_2) = f(x_1) - f(x_2), x_1, x_2 \in P$ , and in the sequel this linear functional will always be extended to a linear functional on  $X$ . So we have:  *$B$  is a base for the cone  $P$  if and only if a strictly positive (not necessarily continuous) linear functional  $f$  of  $X$  exists so that,  $B = \{x \in P \mid f(x) = 1\}$ .* Then we say that **the base  $B$  is defined by the functional  $f$**  and we denote it by  $B_f$ . If  $B$  is a base for the cone  $P$  with  $0 \notin \overline{B}$ , then a continuous linear functional  $x^* \in X^*$  separating  $\overline{B}$  and  $0$  exists. Then  $x^*$  is strictly positive and, if  $B$  is bounded, the base for  $P$  defined by  $x^*$  is also bounded. Therefore we can summarize these facts as follows.

*The cone  $P$  has a base defined by a continuous linear functional  $x^*$  of  $X$  if and only if  $P$  has a base  $B$  with  $0 \notin \overline{B}$ . If moreover the base  $B$  is bounded the base for  $P$  defined by  $x^*$  is bounded.*

Moreover it holds the following well-known result.

**Proposition 2.1.** ([13], Theorem 3.8.4). *A cone  $P$  of a normed space  $Y$  has a bounded base  $B$  with  $0 \notin \overline{B}$  if and only if the dual cone  $P^0$  of  $P$  in  $Y^*$  has interior points. Moreover for every  $x^* \in \text{int}P^0$ , the base  $B_{x^*}$  of  $P$  defined by  $x^*$  is bounded.*

Now we recall a notion introduced in [6]. A cone  $P$  is a **mixed based cone** if there exists a strictly positive  $x^* \in P^0$  which is not an interior point of  $P^0$  and  $\text{int}P^0 \neq \emptyset$ , or equivalently if  $P$  has a bounded and an unbounded base defined (the bases for  $P$ ) by continuous linear functionals.

Let  $Y$  and  $Z$  be two normed spaces. The cone  $P \subseteq Y$  is **isomorphic** to the cone  $K \subseteq Z$  if there exists an additive, positively homogeneous and one-to-one map  $T$  of  $P$  onto  $K$  such that  $T$  and  $T^{-1}$  are continuous in the induced topologies. Then we also say that  $T$  is an isomorphism of  $P$  onto  $K$  and that  $P$  is **embeddable** in  $Z$ .

We close this section by a known result, useful in this article. Recall that if  $\{x_n\}$  is a Schauder basis of  $X$ , then

$$P = \left\{ \sum_{i=1}^{\infty} \lambda_i x_i \mid \lambda_i \geq 0, \text{ for any } i \right\},$$

is **the positive cone of the basis**  $\{x_n\}$ . A sequence  $\{x_n\}$  of an ordered Banach space is a **positive basis** of  $X$  if it is a Schauder basis of  $X$  and the positive cone  $X_+$  of  $X$  and the positive cone of the basis  $\{x_n\}$  coincide.

The usual bases of the spaces  $\ell_p$ ,  $1 \leq p < \infty$  and the space  $c_0$  are simple examples of positive bases.

The next easy proposition has been used in [18] and we present it here because it is useful in our study.

**Proposition 2.2.** *Let  $X$  be an infinite dimensional Banach space with a normalized Schauder basis  $\{x_n\}$  and suppose that  $\{x_n^*\}$  is the sequence of the coefficient functionals of  $\{x_n\}$ . If  $x^* = \sum_{i=1}^{\infty} \lambda_i x_i^* \in X^*$  with  $\lambda_i > 0$  for each  $i$ , then  $x^*$  is strictly positive on the positive cone  $P$  of the basis  $\{x_n\}$ , and  $x^*$  defines an unbounded base  $B_{x^*}$  on  $P$ .*

Indeed, it is easy to see that  $\lim_{i \rightarrow \infty} \lambda_i = 0$  and that  $\{\frac{x_n}{\lambda_n}\}$  is an unbounded sequence of  $B_{x^*}$ .

### 3 Reflexive cones and a characterization of reflexivity

We start with the notion of reflexive cone.

**Definition 3.1.** A cone  $P$  of a Banach space  $X$  is *reflexive* if the set  $B_X^+ = B_X \cap P$  is weakly compact.

*Remark 3.2.* The following properties of a reflexive cone follow immediately from the definition:

1. A reflexive cone is always closed. Indeed, if  $x_n \in P$  and  $x_n \rightarrow x$ , there exists  $\rho \in \mathbb{R}_+$  so that  $x_n \in \rho B_X^+$  for each  $n$ , therefore  $x \in \rho B_X^+ \subseteq P$ .
2. Any closed cone of a reflexive space is reflexive. The converse does not hold in general but, if a Banach space  $X$  has a reflexive and generating cone  $P$ , then  $X$  is reflexive. Indeed, by [13], Theorem 3.5.2, the cone  $P$  gives an open decomposition of  $X$  and hence the unit ball  $B_X$  is a weakly compact set.

Now let us recall some standard notations. Let  $X$  be a Banach space, we denote by  $J_X : X \rightarrow X^{**}$  the natural embedding of  $X$  in  $X^{**}$ . For the sake of simplicity, for any  $x \in X$  we denote by  $\widehat{x}$  the image  $J_X(x)$  of  $x$  in  $X^{**}$  and for any subset  $A \subseteq X$  we denote by  $\widehat{A}$  the set  $J_X(A) \subseteq X^{**}$ . Of course for any  $x^* \in X^*$ ,  $\widehat{x^*}$  is the natural image  $J_{X^*}(x^*)$  of  $x^*$  in  $X^{***}$  and for any  $C \subseteq X^*$ ,  $\widehat{C}$  is the set  $J_{X^*}(C) \subseteq X^{***}$ . Finally, given a subspace  $V$  in  $X$ , we denote by  $V^\perp$  the annihilator  $V^\perp = \{x^* \in X^* \mid x^*(x) = 0, \text{ for any } x \in V\}$  of  $V$  in  $X^*$ .

The next result shows that a reflexive cone exhibits the same behavior as a reflexive space with respect to the second dual.

**Theorem 3.3.** *A closed cone  $P$  of a Banach space  $X$  is reflexive if and only if*

$$\widehat{P} = P^{00}.$$

*Proof.* Assume that the cone  $P \subseteq X$  is reflexive. Then, the set  $\widehat{B_X^+}$  is a weak\* compact set and therefore, for any  $\alpha > 0$ , the set

$$\widehat{P} \cap \alpha B_{X^{**}} = \widehat{\alpha B_X^+}$$

is a weak\* closed set. By Krein-Smulian Theorem, [15], Theorem 2.7.11,  $\widehat{P}$  is a weak\* closed cone. We thus get  $P^{00} = \widehat{P}$  because as one can easily show  $P^{00}$  is the weak\* closure of  $\widehat{P}$ .

Now we prove the other implication. Let us suppose that the equality  $P^{00} = \widehat{P}$  holds. By Banach-Alaoglu theorem, the set

$$\widehat{P \cap B_X} = \widehat{P} \cap B_{X^{**}} = P^{00} \cap B_{X^{**}}$$

is weak\* compact. Since the map  $J_X$  is a weak-to-relative weak\* homeomorphism from  $X$  onto  $\widehat{X}$ , [15], Proposition 2.6.24, the set  $B_X^+$  is a weakly compact set of  $X$  and, therefore the cone  $P$  is reflexive.  $\square$

The next step toward the characterization of reflexive spaces is the following result where we prove that the third dual of a reflexive cone  $P$  of a Banach space  $X$  can be decomposed in the same way as the third dual space of  $X$  is decomposed by the well known formula, [8], Lemma I.12:

$$X^{***} = \widehat{X^*} \oplus (\widehat{X})^\perp. \quad (1)$$

**Theorem 3.4.** *If  $P$  is a reflexive cone of a Banach space  $X$ , then*

$$P^{000} = \widehat{P^0} + (\widehat{X})^\perp.$$

Moreover, every  $p^{***} \in P^{000}$  has a unique decomposition  $p^{***} = x^{***} + y^{***}$ , where  $x^{***} \in \widehat{P^0}$  and  $y^{***} \in (\widehat{X})^\perp$ .

*Proof.* We will show first that  $P^{000} \subseteq \widehat{P^0} + (\widehat{X})^\perp$ . By formula (1), every  $p^{***} \in P^{000}$  has a unique decomposition

$$p^{***} = x^{***} + y^{***},$$

where  $x^{***} \in \widehat{X^*}$  and  $y^{***} \in (\widehat{X})^\perp$  and suppose that  $x^{***} = \widehat{x^*}$ , where  $x^* \in X^*$ . For every  $p^{**} \in P^{00}$  we have

$$0 \leq p^{***}(p^{**}) = x^{***}(p^{**}) + y^{***}(p^{**}).$$

Since  $P$  is reflexive we have  $P^{00} = \widehat{P}$ , therefore there exist  $p \in P$  so that  $\widehat{p} = p^{**}$ . So we have

$$0 \leq p^{***}(p^{**}) = x^{***}(\widehat{p}) + y^{***}(\widehat{p}) = x^{***}(\widehat{p}) = x^*(p) \quad (2)$$

for every  $p \in P$ . So we have  $x^* \in P^0$  and  $x^{***} \in \widehat{P^0}$ , therefore

$$P^{000} \subseteq \widehat{P^0} + (\widehat{X})^\perp.$$

For the converse suppose that  $p^* \in P^0$  and  $y^{***} \in (\widehat{X})^\perp$ . Since  $P$  is reflexive, every  $p^{**} \in P^{00}$  is of the form  $p^{**} = \widehat{p}$ , where  $p \in P$ , therefore we have

$$(\widehat{p}^* + y^{***})(p^{**}) = (\widehat{p}^* + y^{***})(\widehat{p}) = p^*(p) \geq 0.$$

The last relation implies that

$$\widehat{P^0} + (\widehat{X})^\perp \subseteq P^{000}$$

which completes the proof.  $\square$

**Theorem 3.5.** *A Banach space  $X$  is reflexive if and only if there exists a closed cone  $P$  of  $X$  so that the cones  $P$  and  $P^0$  are reflexive.*

*Proof.* If  $X$  is reflexive the thesis follows immediately. Now let us suppose that there exists a closed cone  $P \subseteq X$  such that  $P$  and  $P^0$  are reflexive. By Theorem 3.4, we have

$$P^{000} = \widehat{P^0} + (\widehat{X})^\perp. \quad (3)$$

Moreover, by Theorem 3.3 it holds

$$P^{000} = \widehat{P^0}. \quad (4)$$

Since the decomposition of every element of  $P^{000}$  is unique, the comparison between (3) and (4) implies that  $(\widehat{X})^\perp = \{0\}$ . From this we conclude that  $X^{**} = \widehat{X}$ , proves the theorem.  $\square$

Theorem 3.5 implies that in every non reflexive Banach space a reflexive cone cannot have a dual cone which is reflexive. The following example shows such a situation.

**Example 3.6.** Let  $X = L_1([0, 1])$ ,  $Y$  is the closed subspace of  $X$  generated by the Rademacher functions  $\{r_n\}$ , and let  $P$  be the positive cone of  $\{r_n\}$ . Recall that  $\{r_n\}$  is a basic sequence in  $L_1([0, 1])$ , equivalent to the standard basis of  $\ell_2$ , therefore  $Y$  is isomorphic to  $\ell_2$  and the cone  $P$  is reflexive. By Theorem 3.5, the dual cone  $P^0$  of  $P$  in  $L_\infty([0, 1])$  is not reflexive.

We underline that the previous example exhibits a reflexive cone  $P$  in a non reflexive space  $X$  where  $\overline{P - P}$  is a reflexive subspace of  $X$ . This does not hold in general as the next example shows. We underline also that in the next example, the subspace  $P - P$  generated by  $P$  is dense in  $X$ .

**Example 3.7.** Suppose that  $X = L_1([0, 1])$  and

$$D = \{d \in L_1([0, 1]) : 0 \leq d \leq \mathbf{1}, \|d\| \geq \frac{1}{2}\}.$$

where  $\leq$  is the usual order of  $L_1([0, 1])$  and  $\mathbf{1} \in L_1([0, 1])$  is the function identically equal to 1. Then  $D$  is a closed, convex and bounded set, therefore the cone  $P$  of  $L_1([0, 1])$  generated by  $D$  is closed. For every  $x \in P \cap B_X$  we have  $x = \lambda d$ ,  $d \in D$  with  $\lambda = \frac{\|x\|}{\|d\|} \leq 2$ , therefore  $P \cap B_X$  is a closed and convex subset of the order interval  $[0, 2\mathbf{1}]$ . Now we recall that each order interval of  $L_1[0, 1]$  is weakly compact because  $X$  has order continuous norm, [3], Theorem 12.9. Therefore the set  $B_X^+ = P \cap B_X$  is weakly compact and hence the cone  $P$  is reflexive. Moreover, we remark that  $\overline{P - P} = X$ . Indeed, if  $\{I_i\}$  is the sequence of subintervals  $I_1 = [0, \frac{1}{2}]$ ,  $I_2 = [\frac{1}{2}, 1]$ ,  $I_3 = [0, \frac{1}{4}]$ ,  $I_4 = [\frac{1}{4}, \frac{1}{2}]$ ,  $I_5 = [\frac{1}{2}, \frac{3}{4}]$ ,  $I_6 = [\frac{3}{4}, 1]$ , ... of  $[0, 1]$ ,  $I'_i$  is the complement of  $I_i$  and  $\mathcal{X}_{I'_i}$  is the characteristic function of  $I'_i$ , then  $\mathcal{X}_{I'_i} \in D$  for any  $i$ . Moreover every element of the Haar basis of  $L_1([0, 1])$  can be written as the difference of two functions of the form  $\mathcal{X}_{I'_i}$ , therefore  $P - P$  is dense in  $X$ . Finally, it is easy to see that the cone  $P$  is normal and the basis for  $P$  defined by the constant function  $\mathbf{1}$  is bounded.

## 4 Bases of reflexive cones

This section is devoted to the study of the reflexive cones that admit a base defined by a continuous linear functional. This class of reflexive cones is a large subset of the whole class of reflexive cones. Nevertheless, there exist some reflexive cones that have not a base defined by a continuous linear functional, as shown by the following example.

**Example 4.1.** Let us consider an uncountable set  $\Gamma$ , then space  $\ell_2(\Gamma)$  endowed with the pointwise order, is a reflexive Banach lattice without strictly positive, continuous linear functionals. Therefore the lattice cone  $\ell_2^+(\Gamma)$  is a reflexive cone without a base defined by a continuous linear functional. Moreover, the same behavior appears in the spaces  $\ell_p(\Gamma)$  with  $1 < p < \infty$ .

We begin by recalling the following **Dichotomy Theorem** about the bases of cones.

**Theorem 4.2.** ([19], Theorem 4). *Suppose that  $\langle X, Y \rangle$  is a dual system. If  $X$  is a normed space,  $P$  is a  $\sigma(X, Y)$ -closed cone of  $X$  so that the positive part  $B_X^+ = B_X \cap P$  of the unit ball  $B_X$  of  $X$  is  $\sigma(X, Y)$ -compact, we have: either every base for  $P$  defined by a vector  $y \in Y$  is bounded or every such base for  $P$  is unbounded.*

In our setting, by Theorem 4.2 we have:

**Theorem 4.3.** *Any reflexive cone of the Banach space is not a mixed based cone.*



The converse of the above theorem does not hold because  $c_0^+$  is not a mixed based cone, Proposition 2.2, but  $c_0^+$  is not reflexive.

We now provide a sufficient condition to ensure that a given cone is reflexive, based on an assumption about the boundedness of the bases of the cone.

**Proposition 4.4.** *Let  $X$  be a Banach space ordered by the closed cone  $P$ . If the set  $P^{0s}$  of strictly positive and continuous linear functionals of  $X$  is nonempty and for any  $x^* \in P^{0s}$  the base  $B_{x^*}$  for  $P$  defined by  $x^*$  is bounded, then the cone  $P$  is reflexive.*

*Proof.* Since every base for  $P$  defined by  $x^* \in X^*$  is bounded we have that  $P^{0s} = \text{int}(P^0)$ . Hence, by Lemma 3.4 in [6], we have that the base  $B_{x^*}$  for  $P$  is weakly compact for every  $x^* \in X^*$ . Now let us fix  $x^* \in P^{0s}$ . Since  $B_{x^*}$  does not contain zero, there exists a positive real number  $\rho$  such that  $\rho B_X \cap B_{x^*} = \emptyset$ . It is easy to check that the set

$$\bigcup_{0 \leq \alpha \leq 1} \alpha B_{x^*},$$

is a weakly compact set which contains the closed set  $\rho B_X \cap P$ . Therefore, the cone  $P$  is reflexive.  $\square$

We underline that the converse of Proposition 4.4 does not hold. Indeed the reflexive cone  $\ell_2^+$  is such that every base  $B_{x^*}$  for  $\ell_2^+$  is unbounded for every strictly positive linear functional  $x^* \in \ell_2$ , Proposition 2.2.

The results proved about the bases of reflexive cones allow us to formulate a characterization of reflexive cones in the framework of the theory of Banach spaces.

Before to state the theorem we recall a known result, that will play a central role in the proof: *Let  $\{x_n\}$  be a sequence in a Banach space  $X$  which is not norm-convergent to 0. If  $\{x_n\}$  is weakly Cauchy and not weakly convergent, then  $\{x_n\}$  has a basic subsequence, [10], Theorem 1.1.10.*

**Theorem 4.5.** *A closed cone  $P$  of a Banach space  $X$  is reflexive if and only if  $P$  does not contain a closed cone isomorphic to the positive cone of  $\ell_1$ .*

*Proof.* Let  $P$  be reflexive. Suppose that  $Q \subseteq P$  is a closed cone isomorphic to  $\ell_1^+$ . Then  $Q$  as a subcone of  $P$  is also reflexive. By Theorem 4.5 in [6],  $Q$  is a mixed based cone which contradicts Theorem 4.3. Hence,  $P$  does not contain a closed cone isomorphic to  $\ell_1^+$ .

To prove the other side of the equivalence, let us suppose that  $P$  does not contain a closed cone isomorphic to  $\ell_1^+$ . Now, on the contrary, suppose that  $P$  is not a reflexive cone. Then  $B_X^+ = B_X \cap P$  is not a weakly compact set, therefore there exists a sequence  $\{x_n\}$  in  $B_X^+$  which does not admit a weakly convergent subsequences. Since  $P$  does not contain a closed cone isomorphic to  $\ell_1^+$ ,  $\ell_1$ - Rosenthal Theorem, [7], ensures that there exists a weakly Cauchy subsequence of  $\{x_n\}$ , which we denote again by  $\{x_n\}$ . By the result mentioned just before this theorem,  $\{x_n\}$  has a basic subsequence which we denote again by  $\{x_n\}$ , for the sake of simplicity. The sequence  $\{x_n\}$  does not have a weakly

convergent subsequences, therefore  $\{x_n\}$  is not weakly convergent to 0. Hence, there exists  $x^* \in X^*$  and a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $x^*(x_{n_k}) \geq 1$  for each  $k \in \mathbb{N}$ . Therefore  $\{x_{n_k}\}$  is a basic sequence of  $\ell_+$ -type and the cone

$$K = \left\{ p \in X : p = \sum_{k=1}^{\infty} \alpha_k x_{n_k}; \alpha_k \geq 0 \text{ for every } k \in \mathbb{N} \right\} \subseteq P$$

generated by  $\{x_{n_k}\}$  is isomorphic to  $\ell_+^1$ , [21], Theorem 10.2. This contradicts the facts that  $P$  does not contain a cone isomorphic to  $\ell_1^+$ , and the proof is complete.  $\square$

The previous result shows that the impossibility to embed the cone  $\ell_1^+$  in a closed cone  $P$  of a Banach space is equivalent to  $P$  be reflexive. Hence, it is interesting to know whether a cone is isomorphic to  $\ell_1^+$  or not. A detailed study about this topic can be found in [17]. Moreover, Theorem 4.5 yields an interesting corollary that says that two isomorphic cone are reflexive whenever one of them is reflexive.

**Corollary 4.6.** *If the closed cones  $P \subseteq X$ ,  $Q \subseteq Y$  of the Banach spaces  $X, Y$  are isomorphic we have:  $P$  is reflexive if and only if  $Q$  is reflexive.*

*Proof.* Let  $T$  be an isomorphism of  $P$  onto  $Q$ . If we suppose that  $P$  reflexive we have that  $Q$  is also reflexive as follows: If  $Q$  is nonreflexive, then  $Q$  contains a closed cone  $R$  isomorphic to  $\ell_1^+$ , therefore  $T^{-1}(R)$  is a closed cone of  $P$  isomorphic to  $\ell_1^+$ , a contradiction.  $\square$

The following two results concern the inner structure of a reflexive cone under the assumption that either bounded or unbounded base defined by a continuous functional exists.

**Theorem 4.7.** *Suppose that  $P$  is a reflexive cone of a Banach space  $X$ . If  $P$  has a bounded base defined by  $x^* \in X^*$ , then  $P$  does not contain a basic sequence.*

*Proof.* Let  $\{x_n\} \subseteq P$  be a basic sequence. Since the sequence  $y_n = \frac{x_n}{x^*(x_n)}$  is a basic sequence with  $x^*(y_n) = 1$  for each  $n$ , Theorem 10.2 in [21] shows that  $\{y_n\}$  is a basic sequence of  $\ell_+$ -type, hence the cone

$$K = \left\{ p \in X : p = \sum_{n=1}^{\infty} \alpha_n y_n; \alpha_n \geq 0 \text{ for every } n \in \mathbb{N} \right\} \subseteq P$$

generated by  $\{y_n\}$  is isomorphic to  $\ell_1^+$  which is a contradiction.  $\square$

**Theorem 4.8.** *Suppose that  $P$  is a reflexive cone of a Banach space  $X$ . If  $P$  has an unbounded base defined by  $x^* \in X^*$ , then  $P$  contains a normalized basic sequence  $\{x_n\}$  which converges weakly to zero.*

*Proof.* By our assumption that  $P$  has an unbounded base  $B_{x^*}$  there exists a sequence  $\{y_n\}$  such that  $y_n \in B_{x^*}$  for every  $n$  and  $\|y_n\| \rightarrow \infty$ . Since  $P$  is reflexive, the normalized sequence  $x_n = \frac{y_n}{\|y_n\|}$  has a weakly convergent subsequence which we denote again by  $\{x_n\}$  for the sake of simplicity. We claim that  $\{x_n\}$  converges weakly to zero. Indeed, let  $\bar{x}$  be the weak limit of  $\{x_n\}$ . Then  $\bar{x} \in P$  and

$$x^*(\bar{x}) = \lim_{n \rightarrow \infty} x^*(x_n) = \lim_{n \rightarrow \infty} \frac{1}{\|y_n\|} = 0,$$

hence we have  $\bar{x} = 0$  because  $x^*$  is strictly positive on  $P$ . We conclude the proof by applying the well-known Bessaga-Pelczynski Selection Principle (see, e.g., [7]), which implies that  $\{x_n\}$  has a basic subsequence.  $\square$

Now we examine, in Banach spaces, the relations between the existence of a reflexive cone with an unbounded base and the existence of a reflexive subspace. First we note that the existence of a reflexive subspace implies the existence of a reflexive cone with an unbounded base defined by a continuous linear functional. Indeed, if  $V$  is an infinite dimensional reflexive subspace of  $X$ , then the cone  $P$  generated by a basic sequence  $\{v_n\} \subseteq V$  is reflexive and by Proposition 2.2,  $P$  has an unbounded base defined by a vector  $x^* \in X^*$ . In the special case where  $X$  has an unconditional basis we prove below that the converse is also true. For this proof we use Theorem 4.8 and the next result by Bessaga-Pelczynski: *If a Banach space  $X$  has an unconditional basis, then every normalized weakly null sequence of  $X$  contains an unconditional basic sequence*, [15], Theorem 4.3.19.

**Theorem 4.9.** *Let  $X$  be a Banach space with an unconditional basis. If  $P$  is a reflexive cone with an unbounded base defined by a vector of  $X^*$ , then  $\overline{P - P}$  contains an infinite dimensional reflexive subspace.*

*Proof.* Suppose that  $P$  is a reflexive cone, with an unbounded base defined by  $x^* \in X^*$ . By Theorem 4.8,  $P$  contains a normalized weakly null sequence. By the Bessaga-Pelczynski theorem mentioned above,  $P$  contains an unconditional basic sequence  $\{x_n\}$ . Let  $K$  be the cone generated by  $\{x_n\}$ . It is evident that  $K \subseteq P$  is reflexive. By [21], Theorem 16.3,  $K$  is generating in the closed subspace  $Y$  of  $X$  generated by  $\{x_n\}$ , i.e.  $Y = \overline{\text{span}}\{x_n\} = K - K$ . Hence, the cone  $K$  gives an open decomposition in  $Y$ , [13], Theorem 3.5.2, therefore  $Y$  is reflexive subspace of  $\overline{P - P}$ .  $\square$

It is known that both  $c_0$  and  $\ell_1$  have an unconditional basis but they do not contain an infinite dimensional reflexive subspace. Therefore we have

**Corollary 4.10.** *If  $X = c_0$  or  $X = \ell_1$ , then  $X$  does not contain a reflexive cone  $P$  with an unbounded base defined by a vector of  $X^*$ .*

## 5 Strongly reflexive cones

In this section we restrict our attention to a subclass of the set of reflexive cones. Namely, we deal with those cones  $P$  such that the positive part of the unit ball  $B_X^+$  is a compact set.

**Definition 5.1.** A cone  $P$  of a Banach space  $X$  is *strongly reflexive*, if the set  $B_X^+ = B_X \cap P$  is norm compact.

*Remark 5.2.* Suppose that  $P$  is a strongly reflexive cone of an infinite dimensional Banach space  $X$ . Then we have:

1.  $P$  is not generating, i.e  $X \neq P - P$ . Indeed, if we suppose that  $P$  is generating, by [13], Theorem 3.5.2, we have that  $P$  gives an open decomposition of  $X$ , therefore the unit ball  $B_X$  is compact and  $X$  is a finite dimensional space.
2.  $P$  does not have interior points. Indeed, if  $P$  has an interior point, then  $P$  is generating.

We start our study of strongly reflexive cones by recalling a known result proved by Klee in [14] (see also in [1], Theorem II.2.6).

**Theorem 5.3.** A pointed cone  $P$  in a locally convex Hausdorff space has a compact neighborhood of zero, if and only if  $P$  has a compact base.

We remark that the previous theorem can be adapted to our setting as follows: A pointed cone  $P$  in a Banach space is strongly reflexive if and only if  $P$  has a norm compact base.

Now we state two elementary properties of strongly reflexive cones, based on the above theorem of Klee.

**Proposition 5.4.** If  $P$  is a strongly reflexive and pointed cone of a Banach space  $X$ , then

1.  $P$  has a base defined by a vector of  $X^*$  and any such a base for  $P$  is compact;
2.  $P$  is normal.

*Proof.* Suppose that  $P$  is a strongly reflexive and pointed cone of  $X$ . Then by the above theorem,  $P$  has a compact base  $B$ . Hence there exists  $x^* \in X^*$  that separates 0 and  $B$  with  $x^*(x) > 0$  for every  $x \in B$  and it is easy to show that the base  $B_{x^*}$  of  $P$  defined by  $x^*$  is closed and bounded. Thus  $B_{x^*}$  is compact because it is contained in a positive multiple of  $B_X^+$ . Since  $P$  cannot be a mixed based cone, Theorem 4.3, every base for  $P$  defined by a vector  $X^*$  is also bounded. By [13], Proposition 3.8.2,  $P$  is normal because  $P$  has a bounded base. (With respect to the notations used in [13], we underline that well-based cone means cone with a bounded base  $B$  so that  $0 \notin \overline{B}$  and self-allied cone means normal cone).  $\square$

In the next theorem we give a **method of construction of a reflexive cone by a weakly convergent sequence**. By this, we have that any infinite dimensional Banach space  $X$ , contains an infinite dimensional reflexive cone with a bounded base defined by a vector of  $X^*$ .

**Theorem 5.5.** *Let  $\{x_n\}$  be a sequence of a Banach space  $X$  which converges weakly to zero,  $x_0 \notin \overline{\text{co}}\{x_n\}$  and  $D = \overline{\text{co}}\{x_n\} - x_0$ . If  $P = \text{cone}(D)$  is the cone of  $X$  generated by  $D$ , then  $P$  is a reflexive cone with a bounded base defined by a vector of  $X^*$ . Moreover the following statement hold:*

1. *If  $\|x_n\| \geq \delta > 0$  for each  $n$ , then  $P$  is not strongly reflexive;*
2. *If  $\lim_n \|x_n\| = 0$  then  $P$  is strongly reflexive.*

*Proof.* The set of elements of the sequence  $\{x_n\}$  is relatively weakly compact and by the Krein-Smulian Theorem, [15], Theorem 2.8.14,  $\overline{\text{co}}\{x_n\}$  is weakly compact. Therefore  $D$  is a convex, bounded and weakly compact set such that  $0 \notin D$ . The cone  $P$  generated by  $D$  is closed because  $D$  is closed and bounded. Let  $0 < \rho < m = \inf\{\|y\| \mid y \in D\}$ . Then

$$P \cap \rho B_X \subseteq \bigcup_{0 \leq \lambda \leq 1} \lambda D,$$

therefore  $P \cap \rho B_X$  is weakly compact and the cone  $P$  is reflexive. Since  $0 \notin D$ , there exists  $x^* \in X^*$  such that  $x^*$  separates  $D$  and  $0$  and  $x^*(x) > 0$  for every  $x \in D$ . It is easy to check that  $x^*$  defines a bounded base for the cone  $P$ .

If we suppose that  $\|x_n\| \geq \delta > 0$  for each  $n$ , the set  $P \cap \rho B_X$  is not norm compact, because  $\{x_n - x_0\}$  does not have a norm convergent subsequence. Therefore  $P$  is not strongly reflexive and statement 1. is true.

If we suppose that  $\lim_n \|x_n\| = 0$ , then in the above proof the set  $D$  is compact and we have that statement 2. is also true.  $\square$

We give now a characterization of the Banach spaces  $X$  with the Schur property. Recall that  $X$  has the **Schur property** if every weakly convergent sequence of  $X$  is norm convergent. If  $X$  has the Schur property it is clear that any reflexive cone of  $X$  is strongly reflexive, because every weakly compact subset of  $X$  is norm compact. Therefore the interesting part of the next characterization is the converse.

**Theorem 5.6.** *A Banach space  $X$  has the Schur property if and only if every reflexive cone of  $X$  with a bounded base defined by a vector of  $X^*$ , is strongly reflexive.*

*Proof.* For the converse suppose that  $X$  does not have the Schur property. Then  $X$  has a normalized sequence  $\{x_n\}$  with  $x_n \xrightarrow{w} 0$  and by Theorem 5.5, statement 1, we have a contradiction.  $\square$

Now we give below two examples of strongly reflexive cones. The first is a general example. Specifically we give a way to build a strongly reflexive cone in the positive cone  $X_+$  of any Banach lattice  $X$  with a positive basis.

**Theorem 5.7.** *If  $X$  is an infinite dimensional Banach lattice with a positive Schauder basis  $\{e_i\}$ , then  $X_+$  contains a strongly reflexive cone  $P$  so that  $\overline{P - P} = X$ .*

*Proof.* Without loss of generality for our proof, we can suppose that  $\{e_i\}$  is a normalized basis of  $X$ . Consider the cone

$$P = \{x = \sum_{i=1}^{\infty} x_i e_i : 0 \leq x_i \leq \alpha x_{i-1}, \text{ for each } i > 1\},$$

where  $\alpha \in (0, 1)$  is a fixed real number. It is straightforward to see that  $P$  is a closed subcone of  $X_+$ . Also it is easy to show that

$$y = \sum_{i=1}^{\infty} \alpha^{i-1} e_i \in P.$$

Moreover we can show that for every  $x \in P$  we have  $x_i \leq \alpha^{i-1} x_1$  for every  $i > 1$  and also that

$$P = \{x \in X \mid 0 \leq x \leq x_1 y\}, \quad (5)$$

where  $\leq$  is the order of  $X$ . By the positivity of the basis  $\{e_i\}$ , for every  $x \in P$  we have

$$0 \leq x_1 e_1 \leq x, \quad (6)$$

therefore

$$x_1 \|e_1\| \leq \|x\|, \quad (7)$$

because  $X$  is a Banach lattice. But  $\|e_1\| = 1$  because the basis  $\{e_i\}$  is normalized and by (5), (6) and (7) we obtain

$$0 \leq x \leq x_1 y \leq \|x\| y, \quad (8)$$

for every  $x \in P$ . From this we deduce that  $P \cap B_X \subseteq [0, y]$ . By [21], Theorem 16.3, every Banach lattice with a positive Schauder basis has compact order intervals, therefore the order interval  $[0, y]$  is compact and the cone  $P$  is strongly reflexive. Also  $P$  is pointed because it is contained in  $X_+$ .

We shall show that  $\overline{P - P} = X$ . We remark that  $e_1 \in P$  and also that

$$e_i = \frac{1}{\alpha^{i-1}} (e_1 + \alpha e_2 + \dots + \alpha^{i-1} e_i) - \frac{1}{\alpha^{i-1}} (e_1 + \alpha e_2 + \dots + \alpha^{i-2} e_{i-1}) \in P - P,$$

for every  $i \geq 2$ . Therefore  $\overline{P - P} = X$ .  $\square$

By Megginson, Theorem 4.2.22, we have: *if  $\{x_n\}$  is an unconditional basis of a Banach space  $X$ , then  $X$ , ordered by the positive cone of the basis  $\{x_n\}$  is a Banach lattice (under an equivalent norm), therefore we have the following corollary:*

**Corollary 5.8.** *If  $\{x_n\}$  is an unconditional basic sequence of a Banach space  $X$ , then the positive cone of the basis  $\{x_n\}$  contains a strongly reflexive cone  $P$  so that  $\overline{P - P} = Y$ , where  $Y$  is the subspace generated by the basis  $\{x_n\}$ .*

Now we give another example of a strongly reflexive cone  $P$  of the space  $L_1[0, 1]$  such that  $P \subseteq L_1^+[0, 1]$  and  $\overline{P - P} = L_1[0, 1]$ .

**Example 5.9.** Let  $I_1 = [0, 1]$ ,  $I_2 = [0, \frac{1}{2})$ ,  $I_3 = [\frac{1}{2}, 1]$ ,  $I_4 = [0, \frac{1}{4})$ ,  $I_5 = [\frac{1}{4}, \frac{1}{2})$ ,  $I_6 = [\frac{1}{2}, \frac{3}{4})$ ,  $I_7 = [\frac{3}{4}, 1]$ , ... be a sequence of subintervals of  $[0, 1]$ .

Suppose that  $X = L_1([0, 1])$  and  $T : \ell_1 \rightarrow X$  so that

$$T(\xi) = \sum_{i=1}^{\infty} \xi_i \mathcal{X}_{I_i},$$

where  $\mathcal{X}_{I_i}$  is the characteristic function of  $I_i$  and  $\xi = (\xi_i) \in \ell_1$ . Then  $T$  is linear and continuous with  $\|T(\xi)\| \leq \|\xi\|$ . Let us consider  $\eta = (1, \alpha, \alpha^2, \alpha^3, \dots) \in \ell_1^+$  where  $\alpha \in (0, 1)$  and

$$K = \{\xi = (\xi_i) \in \ell_1^+ \mid \xi_i \leq \xi_1 \alpha^{i-1}, \text{ for each } i\}.$$

Then

$$K = \{\xi = (\xi_i) \in \ell_1^+ \mid \xi \leq \xi_1 \eta\},$$

and  $K$  is the strongly reflexive cone of Theorem 5.7 for  $X = \ell_1$ . Suppose that  $Q = T(K)$  and that  $P$  is the closure of  $Q$  in  $L_1([0, 1])$ . For every  $\xi \in K$  we have

$$\xi_1 \mathcal{X}_{I_1} \leq T(\xi) \leq \xi_1 T(\eta),$$

where  $\leq$  is the usual order of  $L_1([0, 1])$ . Therefore we have

$$\xi_1 \leq \|T(\xi)\| \leq \xi_1 \|T(\eta)\|.$$

Now let  $x \in P$ . Then there exists a sequence  $\{\xi^n\} \subseteq K$  such that  $x = \lim_{n \rightarrow \infty} T(\xi^n)$ . Let  $M = 2\|x\|$ . Then we can choose the sequence  $\{\xi^n\}$  such that  $\|T(\xi^n)\| \leq M$  for each  $n$ . Therefore we have

$$\xi_1^n \leq \|T(\xi^n)\| \leq 2\|x\|,$$

for each  $n$ . If we suppose that  $\|x\| \leq 1$  we have  $\xi_1^n \leq 2$ , therefore  $\xi^n \leq 2\eta$ , hence

$$T(\xi^n) \in D = T([0, 2\eta]),$$

for each  $n$ . Since the order interval  $[0, 2\eta]$  of  $\ell_1$  is compact, we have that  $D$  is compact and  $x \in D$ . So we have that  $P \cap B_X \subseteq D$  is a compact set and the cone  $P$  is strongly reflexive. Finally, we remark that  $\mathcal{X}_{I_1} = T(e_1)$ ,  $\mathcal{X}_{I_2} = \frac{1}{\alpha}(T(e_1 + \alpha e_2) - T(e_1))$  and continuing we have that  $\mathcal{X}_{I_i} \in P - P$  for any  $i$ . Therefore  $P - P$  is dense in  $X$ , because the Haar basis of  $L_1[0, 1]$  is consisting by differences of the functions  $\mathcal{X}_{I_i}$ , see in [15], Example 4.1.27.

We conclude this section dealing with some properties about positive operators between ordered Banach spaces.

**Theorem 5.10.** *Suppose that  $E$  and  $X$  are Banach spaces ordered by the closed and pointed cones  $P$  and  $Q$ . If the cone  $P$  gives an open decomposition in  $E$  and  $Q$  is a reflexive (respectively, strongly reflexive) cone of  $X$ , then any positive, linear operator  $T : E \rightarrow X$  is weakly compact (respectively, compact).*

*Proof.* Suppose that  $T : E \longrightarrow X$  is positive, linear operator. By [5], Theorem 2.32,  $T$  is continuous. The set  $T(B_E^+) \subseteq Q$ , is convex and bounded, therefore relatively weakly compact, because  $Q$  is reflexive. By our assumptions,  $W = B_E^+ - B_E^+$  is a neighborhood of zero. The set  $\overline{T(B_E^+)}$  is weakly compact, hence the set  $\overline{T(W)} \subseteq \overline{T(B_E^+) - T(B_E^+)}$  is weakly compact and therefore  $T$  is weakly compact. The case of compact operator is analogous.  $\square$

If  $E$  is a Banach lattice, then  $E_+$  gives an open decomposition in  $E$ , therefore by the above theorem we have:

**Corollary 5.11.** *Any positive, linear operator from a Banach lattice  $E$  into a Banach space  $X$  ordered by a pointed, reflexive (respectively, strongly reflexive) cone is weakly compact (respectively, compact).*

## 6 Reflexive cones and Schauder bases

In this section we study the reflexivity of the positive cone of a Schauder basis of  $X$  in terms of the properties of the basis. Our study has been inspired by the classical results of James in [12]. Suppose that  $X$  is a Banach space with a Schauder basis  $\{x_n\}$  and let  $\{x_n^*\}$  be the sequence of the coefficient functionals of  $\{x_n\}$ . Throughout the whole section we will denote by  $P$ , the positive cone of the basis  $\{x_n\}$ , i.e.  $P = \{x = \sum_{i=1}^{\infty} \lambda_i x_i \mid \lambda_i \geq 0, \text{ for each } i\}$  and by  $Y$  the closed subspace of  $X^*$  generated by  $\{x_n^*\}$ . Note that  $\{x_n^*\}$  is a basic sequence in  $X^*$ . We will also denote by  $Q = \{f = \sum_{i=1}^{\infty} \lambda_i x_i^* \mid \lambda_i \geq 0, \text{ for each } i\}$ , the positive cone of the sequence  $\{x_n^*\}$ , by  $Q^0$  the dual cone of  $Q$  in  $Y^*$  and by  $\Psi(x)$  the restriction of  $\widehat{x}$  to  $Y$ .

We say that  $\{x_n\}$  is **boundedly complete on  $P$**  if for each sequence  $\{a_n\}$  of nonnegative real numbers  $\sup_{n \in \mathbb{N}} \{\|\sum_{i=1}^n a_i x_i\|\} < +\infty$  implies that  $\sum_{i=1}^{\infty} a_i x_i \in P$ . This notion has been defined in [20] for the study of special properties of cones. A simple example of a basis which is boundedly complete on  $P$  but not boundedly complete on the whole space  $X$ , is the summing basis of  $c_0$ .

In the next proposition we give a characterization of Schauder bases which are boundedly complete on the cone  $P$ .

**Proposition 6.1.**  *$\{x_n\}$  is boundedly complete on  $P$  if and only if  $\Psi(P) = Q^0$ .*

*Proof.* Suppose that  $\{x_n\}$  is boundedly complete on  $P$ . We shall show that for any  $y^{**} \in Q^0$ ,  $y^{**} = \Psi(x)$  for some  $x \in P$ .

First we will show that  $x = \sum_{n=1}^{\infty} y^{**}(x_n^*) x_n$  exists. This will imply that  $y^{**} = \Psi(x)$  because  $y^{**}(x_i^*) = \widehat{x}(x_i^*)$ , for each  $i$ .

For each  $m \in \mathbb{N}$  and each  $z^* = \sum_{n=1}^{\infty} a_n x_n^* \in Y$  we have

$$|(\sum_{n=1}^m y^{**}(x_n^*) \widehat{x_n})(z^*)| = |y^{**}(\sum_{n=1}^m a_n x_n^*)| \leq M^* \|y^{**}\| \|z^*\|,$$



where  $M^*$  is the basis constant of the basic sequence  $\{x_n^*\}$ , therefore

$$\left\| \sum_{n=1}^m y^{**}(x_n^*) \widehat{x_n} \right\| \leq M^* \|y^{**}\|$$

where  $\sum_{n=1}^m y^{**}(x_n^*) \widehat{x_n}$  is considered as a functional of  $Y$  and its norm is considered on  $Y$ . By [15], Lemma 4.4.3, we have:

$$\left\| \sum_{n=1}^m y^{**}(x_n^*) \widehat{x_n} \right\| \leq M M^* \|y^{**}\|,$$

where  $M$  is the basis constant of the basis  $\{x_n\}$ .

Therefore  $\left\| \sum_{n=1}^m y^{**}(x_n^*) x_n \right\| \leq M M^* \|y^{**}\|$  for each  $m \in \mathbb{N}$ , hence  $x = \sum_{n=1}^\infty y^{**}(x_n^*) x_n \in P$  because  $\{x_n\}$  is boundedly complete on  $P$ . Therefore,  $\Psi(x) = y^{**}$  and  $\Psi(P) = Q^0$ .

For the converse suppose that  $\Psi(P) = Q^0$  and suppose also that  $\{a_n\}$  is a sequence of nonnegative real numbers with  $\left\| \sum_{i=1}^n a_i x_i \right\| \leq M$  for each  $n \in \mathbb{N}$ . Then  $s_n = \sum_{i=1}^n a_i \Psi(x_i)$  is also bounded in  $Y^*$ . By Alaoglu's Theorem, there exists a subnet  $\{s_{n_a}\}_{a \in A}$  of  $\{s_n\}$  such that

$$s_{n_a} \xrightarrow{w^*} y^* \in Q^0 = \Psi(P).$$

Therefore  $y^* = \sum_{i=1}^\infty y^*(x_i^*) \Psi(x_i)$ , in the weak-star topology of  $Y^*$ . Since  $\Psi(P) = Q^0$ , there exists  $x \in P$  so that we have that  $y^* = \Psi(x)$ . Since  $\{x_n\}$  is a basis of  $X$  we have that  $x = \sum_{i=1}^\infty x_i^*(x) x_i$ . For any  $i$  we have

$$x_i^*(x) = \Psi(x)(x_i^*) = y^*(x_i^*) = \lim s_{n_a}(x_i^*) = a_i, \text{ for each } i.$$

Hence  $x = \sum_{i=1}^\infty a_i x_i$  and  $\{x_n\}$  is boundedly complete on  $P$ .  $\square$

The next results show the link between reflexivity of  $P$  and the boundedly completeness of the basis  $\{x_n\}$  on the cone  $P$ .

**Theorem 6.2.** *If the positive cone  $P$  of the Schauder basis  $\{x_n\}$  of the Banach space  $X$  is reflexive, then  $\{x_n\}$  is boundedly complete on  $P$ .*

*Proof.* Suppose that  $\{a_n\}$  is a sequence of nonnegative real numbers and suppose that  $s_n = \sum_{i=1}^n a_i x_i$  such that  $\|s_n\| \leq M$  for each  $n \in \mathbb{N}$ . Since the cone  $P$  is reflexive, there exists a subsequence  $\{s_{k_n}\}$  such that  $s_{k_n} \xrightarrow{w} x$ , where  $x = \sum_{i=1}^\infty x_i^*(x) x_i$ . Then  $x_i^*(x) = \lim_{n \rightarrow \infty} x_i^*(s_{k_n}) = a_i$ , for each  $i$ , therefore  $x = \sum_{i=1}^\infty a_i x_i \in P$  and  $\{x_n\}$  is boundedly complete on  $P$ .  $\square$

Recall that the basis  $\{x_n\}$  of  $X$  is **shrinking** if  $\{x_n^*\}$  is a basis of  $X^*$ . It is known, [12], that a Banach space  $X$  with a Schauder basis  $\{x_n\}$  is reflexive if and only if  $\{x_n\}$  is shrinking and boundedly complete (on  $X$ ). In the next theorem, under the weaker assumption that  $\{x_n\}$  is boundedly complete on  $P$ , we prove that the cone  $P$  is reflexive. The assumption that the shrinking basis  $\{x_n\}$  is boundedly complete only on the cone  $P$ , is not enough to ensure the reflexivity of the whole space  $X$ , see Example 6.4 below.

**Theorem 6.3.** *If the Schauder basis  $\{x_n\}$  of the Banach space  $X$  is shrinking and boundedly complete on  $P$ , then  $P$  is reflexive.*

*Proof.* We will show that  $\widehat{P} = P^{00}$ . For any  $x^{**} \in P^{00}$  we have that  $x^{**} = \sum_{i=1}^{\infty} x^{**}(x_i^*) \widehat{x}_i$ , in the weak-star topology of  $X^{**}$ , because the basis  $\{x_n\}$  is shrinking. Therefore we have

$$s_n = \sum_{i=1}^n x^{**}(x_i^*) \widehat{x}_i \xrightarrow{w^*} x^{**}.$$

Hence there exists  $M > 0$ , such that  $\|s_n\| = \|\sum_{i=1}^n x^{**}(x_i^*) x_i\| \leq M$ , for each  $n$ . Also  $x^{**}(x_i^*) \geq 0$  for each  $i$ , because  $x^{**} \in P^{00}$ , therefore

$$x = \sum_{i=1}^{\infty} x^{**}(x_i^*) x_i \in P,$$

because  $\{x_n\}$  is boundedly complete on  $P$  and by the uniqueness of the weak-star limit we have  $x^{**} = \widehat{x} \in \widehat{P}$ , therefore  $P$  is reflexive.  $\square$

We end this section with an example of a nonreflexive space  $X$  with a shrinking Schauder basis which is boundedly complete on its positive cone  $P$  but not boundedly complete on the whole space  $X$ .

**Example 6.4.** Let  $J$  be the James space, i.e. the space of all real sequences  $\{\alpha_n\}$  such that  $\lim \alpha_n = 0$  and

$$\sup \left\{ \sum_{j=1}^{n-1} (\alpha_{k_j} - \alpha_{k_{j+1}})^2 \right\}^{1/2} < +\infty$$

where the supremum is taken over all  $n \in \mathbb{N}$  and all finite increasing sequences  $k_1 < k_2 < \dots < k_n$  in  $\mathbb{N}$ . The norm  $\|\{\alpha_n\}\|_J$  is defined to be this supremum. It is well known that the standard unit vector  $\{e_n\}$  in  $J$  is a monotone shrinking basis (see [8] for a general reference of James space). Of course such a basis is not boundedly complete since  $J$  is not a reflexive space. It is easy to check that  $\{e_n\}$  is not boundedly complete on its positive cone.

Also by the formula  $(a - b)^2 \leq 2a^2 + 2b^2$ , for any  $a, b \in \mathbb{R}$  we have that any real sequence  $\{a_i\}$  of  $\ell_2$  belongs to  $J$  with

$$\|\{a_i\}\|_J \leq \left( 2 \sum_{i=1}^{\infty} a_i^2 \right)^{\frac{1}{2}}.$$

Let us consider the sequence  $x_n = (-1)^{n+1} e_n$ . It is clear that  $\{x_n\}$  is again a shrinking basis of  $J$ . We will show that  $\{x_n\}$  is boundedly complete on its positive cone  $P = \{x = \sum_{i=1}^{\infty} \lambda_i x_i \mid \lambda_i \geq 0, \text{ for each } i\}$ .

So we suppose that  $\{\lambda_n\}$  is a sequence of nonnegative real numbers so that  $\sup_n \|\sum_{i=1}^n \lambda_i x_i\|_J \leq M$ .

Suppose that  $a_i = \lambda_i$  if  $i$  is odd and  $a_i = -\lambda_i$  if  $i$  is even.

Then  $\sup_n \left\| \sum_{j=1}^n \alpha_j e_j \right\|_J \leq M$ . Also

$$(a_k - a_{k+1})^2 = (|a_k| + |a_{k+1}|)^2 \geq a_k^2,$$

if  $k$  is odd and

$$(a_k - a_{k+1})^2 = (-|a_k| - |a_{k+1}|)^2 \geq a_k^2,$$

if  $k$  is even, therefore

$$0 \leq \sum_{k=1}^n a_k^2 \leq \sum_{k=1}^n (a_k - a_{k+1})^2 \leq M^2.$$

Therefore the sequence  $\{\alpha_n\} \in \ell_2$ , hence

$$\sum_{i=1}^{\infty} \alpha_i e_i = \sum_{i=1}^{\infty} \lambda_i x_i,$$

exists in  $J$  and the basis  $\{x_n\}$  is boundedly complete on  $P$ . By Theorem 6.3, we have that the cone  $P$  is reflexive. Finally note that  $J = \overline{P - P}$ .

The previous example together with Theorem 4.9 suggest the following unanswered question.

**Problem 6.5.** If  $P$  is a reflexive cone of a Banach space  $X$  with an unbounded base defined by a vector of  $X^*$ , does the subspace  $\overline{P - P}$  contain a reflexive subspace?

## 7 Spaces ordered by reflexive cones

We start this section by proving that  $X$ , ordered by a normal and reflexive cone is **Dedekind complete**, i.e. any increasing net of  $X$ , bounded from above, has a supremum. Recall that a Dedekind complete ordered space is not necessarily a lattice.

**Theorem 7.1.** *Any Banach space  $X$ , ordered by a reflexive and normal cone  $P$ , is Dedekind complete.*

*Proof.* Suppose that  $(x_a)_{a \in A}$  is a increasing net of  $X$ , such that  $x_a \leq x$  for each  $a \in A$ . It is enough to show that  $\sup\{x_a \mid a \geq a_0\}$ , for some fixed  $a_0$ , exists. For each  $a \geq a_0$  we have that  $0 \leq x_a - x_{a_0} \leq x - x_{a_0}$ , hence  $\|x_a - x_{a_0}\| \leq M$  for each  $a \geq a_0$ , because  $P$  is normal. For any  $x^* \in P^0$ ,  $x^*(x_a - x_{a_0})$  is increasing and bounded by  $M\|x^*\|$ , therefore  $\lim_a x^*(x_a - x_{a_0})$  exists and we denote this limit by  $f(x^*)$ , i.e.  $f(x^*) = \lim x^*(x_a - x_{a_0})$ . Note that the cone  $P^0$  is generating in  $X^*$  because  $P$  is normal, [5], Theorem 2.26, therefore any  $x^* \in X^*$ , has a decomposition of the form  $x^* = x_1^* - x_2^*$  and we define  $f(x^*) = f(x_1^*) - f(x_2^*)$ . It is easy to show that  $f$  is well defined and that  $f(x^*) = \lim x^*(x_a - x_{a_0})$ . Then

$f$  is a continuous linear functional of  $X^*$  and  $f \in P^{00}$ . By the reflexivity of  $P$  we have that  $\hat{P} = P^{00}$ , therefore there exists  $y \in P$  such that  $f(x^*) = x^*(y)$  for any  $x^* \in X^*$ . It is easy to see that  $x_a - x_{a_0} \xrightarrow{w} y$ . We shall show that  $\sup_{a \geq a_0} (x_a - x_{a_0}) = y$ . Indeed, for any  $x^* \in P^0$  we have that  $x^*(x_a - x_{a_0}) \leq x^*(y)$ . Since the cone  $P$  is closed we have that  $P$  coincides with the dual cone  $(P^0)_0 = \{x \in X \mid x^*(x) \geq 0, \text{ for any } x^* \in P^0\}$ , of  $P^0$  in  $X$ , therefore we have that  $x_a - x_{a_0} \leq y$ . For any  $w \geq x_a - x_{a_0}$  we have  $x^*(x_a - x_{a_0}) \leq x^*(w)$ , therefore  $x^*(y) \leq x^*(w)$ , for any  $x^* \in P^0$ , hence  $y \leq w$  and the theorem is true.  $\square$

In the following we study the lattice property of Banach spaces ordered by reflexive cones. So we will suppose that our cones are pointed (proper). Note that any cone with a base and also any normal cone of  $X$  are pointed. We start by some notations.

It is known, see in [5], Corollary 2.48, that *any reflexive space  $X$  ordered by a normal and generating cone  $P$  has the Riesz decomposition property (i.e. for each  $x, y, z \in P$  we have:  $x \leq y + z \implies x = x_1 + x_2$ , where  $x_1, x_2 \in P$  with  $x_1 \leq y, x_2 \leq z$ ) if and only if  $X$  is a lattice.*

Since any space that contains a generating, reflexive cone is reflexive, the following result is an immediate translation of the above result in the language of reflexive cones.

**Theorem 7.2.** *A Banach space  $X$  ordered by a normal, generating and reflexive cone  $P$  has the Riesz decomposition property if and only if  $X$  is a lattice.*

Let  $Y$  be a normed space ordered by the pointed cone  $P$ . A point  $x_0 \in P \setminus \{0\}$  is an **extremal point** of  $P$  if for any  $x \in X$  with  $0 \leq x \leq x_0$  implies  $x = \lambda x_0$ . If  $Y$  is a vector lattice and there exists a real constant  $a > 0$  so that  $x, y \in Y, |x| \leq |y|$  implies  $\|x\| \leq a\|y\|$ , then  $Y$  is a **locally solid vector lattice**. We continue by the notions of the continuous projection property. This property was defined in [17] for the study of the extreme structure of the bases for cones. We say that an extremal point  $x_0$  of  $P$  **has (admits) a positive projection** if there exists a linear, continuous, positive projection  $P_{x_0}$  of  $Y$  onto the one dimensional subspace  $[x_0]$  generated by  $x_0$ , so that  $P_{x_0}(x) \leq x$  for any  $x \in P$ . We say that  $Y$  has the **continuous projection property** if it holds:  $x_0 \in Ep(P)$  implies that  $x_0$  admits a positive projection. Also in [17] it is proved that if  $Z = P - P$  and the cone  $P$  is closed, we have:

(i)  *$Y$  has the continuous projection property if and only if  $Z$ , ordered by the cone  $P$ , has the continuous projection property and (ii) if  $Z$  is a locally solid vector lattice then  $Z$  and therefore also  $Y$ , have the continuous projection property.* So the continuous projection property depends on the cone  $P$ , therefore we can also say that the cone  $P$  has the continuous projection property.

As it is noted in [17], if  $Y$  is a Banach space,  $P$  is pointed, closed and generating and  $Y$  has the Riesz decomposition property, then  $Y$  has the continuous projection property, therefore *in a Banach space, ordered by a pointed, closed and generating cone, the continuous projection property is weaker than the Riesz decomposition and also than the lattice property.*

Note that in Example 4.1, of [17], a reflexive, generating and normal cone  $P$  of a reflexive space  $X$  with a bounded base which fails the Riesz decomposition property, is given.

Recall also Theorem 4.1 of [17] which is basic for the study of the lattice property of the reflexive cones: *Let  $P$  be an infinite-dimensional closed cone of a Banach space  $X$ . If  $P$  has the continuous projection property and  $P$  has a closed, bounded base  $B$  with the Krein-Milman property, then  $P$  is isomorphic to the positive cone  $\ell_1^+(\Gamma)$  of some  $\ell_1(\Gamma)$  space.*

**Theorem 7.3.** *If  $P$  is a reflexive cone of a Banach space  $X$  with a closed, bounded base  $B$ , then  $P$  does not contain an infinite dimensional, closed cone  $K$  with the continuous projection property.*

*Proof.* First we remark that the base  $B$  for  $P$  has the Krein-Milman property, i.e. each closed, convex and bounded subset  $C$  of  $B$  is the closed convex hull of its extreme points. Indeed, for any  $C \subseteq B$ , closed and convex, by the Krein-Milman Theorem, we have  $C = \overline{\text{co}}^w \text{ep}(C) = \overline{\text{co}} \text{ep}(C)$ , where  $\text{ext}(C)$  denotes the set of the extreme points of  $C$  and  $\overline{\text{co}}^w \text{ep}(C)$ ,  $\overline{\text{co}} \text{ep}(C)$ , is the closed convex hull of  $\text{ep}(C)$  in the weak, norm, topology of  $X$ . If we suppose that  $K \subseteq P$  is an infinite dimensional, closed cone with the continuous projection property, then by [17], Theorem 4.1,  $K$  is isomorphic to the positive cone  $\ell_1^+(\Gamma)$  of some  $\ell_1(\Gamma)$  space, therefore  $\ell_1^+$  is contained in  $K$  because  $\ell_1^+$  is contained in  $\ell_1^+(\Gamma)$ . This is a contradiction because  $K$  as a reflexive cone cannot contain  $\ell_1^+$ , therefore  $P$  does not contain an infinite dimensional, closed cone  $K$  with the continuous projection property and the theorem is true.  $\square$

If in the above theorem we suppose that  $K \subseteq P$  is a closed cone and the space  $Y = P - P$  is ordered by the cone  $K$ , we have: If  $Y$  is a locally solid vector lattice, then by the above remarks we have that  $P$  has the continuous projection property and by the theorem we have a contradiction. So we have the following corollary:

**Corollary 7.4.** *If  $P$  is a reflexive cone of a Banach space  $X$  with a closed, bounded base  $B$ , then  $P$  does not contain an infinite dimensional closed cone  $K$  so that the space  $Y = P - P$ , ordered by the cone  $K$ , is a locally solid vector lattice.*

If in the above theorem we suppose that  $X$  is a vector lattice, we have again a contradiction because the cone  $P$  has the continuous projection property. So we have the corollary:

**Corollary 7.5.** *Any reflexive and generating cone of an infinite dimensional Banach space  $X$  with a bounded base cannot be a lattice cone.*

*Remark 7.6.* Note that in the Choquet theory, pointed cones  $P$  in locally convex Hausdorff spaces  $X$  with a compact base are studied, see in the book of Alfsen, [1]. If a cone has a compact base, then it has a compact neighborhood of zero. One of the main purposes of the Choquet theory is to give necessary and

sufficient conditions, so that  $X$  ordered by the cone  $P$  to be a lattice. To this end a theory of representation of the vectors of the base by measures supported "close" to the extreme points of the base is developed. A necessary condition, in order  $X$  to be a lattice, is of course the cone  $P$  to be generating. If  $X$  is a Banach space, then any cone  $P$  with a compact base is strongly reflexive and by the fact that  $P$  is generating we have that  $X$  is finite dimensional, therefore the basic problem of the Choquet theory as it is formulated in locally convex Hausdorff spaces, is pointless in Banach spaces because it concerns only finite dimensional spaces.

Moreover, if we suppose that the cone  $P$  has a weakly compact base, then  $P$  is a reflexive cone with a bounded base. By Theorem 7.3 and its corollaries we have: If the cone  $P$  is generating, then  $X$  cannot be a lattice, therefore in the case of a Banach space ordered by a generating, closed cone with a weakly compact base the answer to the basic problem of the Choquet theory is negative. A negative answer is given also by Corollary 7.4, in the case where the space  $Y = P - P$  generated by  $P$  is a locally solid vector lattice.

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